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AFDELING NUMERIEKE WISKUNDE NW 37/77 FEBRUARI
(DEPARTMENT OF NUMERICAL MATHEMATICS)

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ON THE NUMERICAL SOLUTION OF PARABOLIC EQUATIONS IN A SINGLE SPACE VARIABLE BY THE CONTINUOUS TIME GALERKIN METHOD

Preprint

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O).

AMS(MOS) subject classification scheme (1970): 65H10

On the numerical solution of parabolic equations in a single space variable by the continuous time Galerkin method *)

by

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ABSTRACT

The initial boundary value problem is solved numerically by Galerkin's method using continuous piecewise polynomial functions. A variant is developed which yields purely explicit initial value problems with a sparse Jacobian. Also, an alternative proof is given for the pointwise convergence at the knots.

KEY WORDS AND PHRASES: *Parabolic initial boundary value problems in one space variable, continuous time Galerkin, superconvergence, explicit initial value problems.*

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*) This report will be submitted for publication elsewhere

1. INTRODUCTION

We consider the linear initial boundary value problem

$$\frac{\partial u}{\partial t} = -Lu \equiv \frac{\partial}{\partial x}[p(x)\,\frac{\partial u}{\partial x}] - q(x)u;$$

$$(1.1) \qquad x \in [0,1] = I; \quad t \in [0,\infty) = J;$$

$$u(t,0) = u(t,1) = 0;$$
$$u(0,x) = u_0(x),$$

where the functions $p(x)$, $q(x)$ and $u_0(x)$ are supposed to be sufficiently smooth.

In §2 we introduce some notations we need throughout this paper and give a summary of the theory on the continuous time Galerkin method (also called Phaedo-Galerkin method) using continuous piecewise polynomials.

In §3 we prove that, as in the case of the two-point boundary value problems (see DOUGLAS & DUPONT [1974]), at the mesh points the local order of convergence is much higher than the global order of convergence. This phenomenon is called superconvergence.

In §4 we evaluate a variant of the continuous time Galerkin method which yields pure explicit vector initial value problems, without loss of accuracy. In the last § we illustrate the results of §4 by a simple numerical example.

2. THE CONTINUOUS TIME GALERKIN METHOD

2.1 Notations

For any interval $E \subset I$, we introduce the Sobolev space $H^m(E)$, $m \geq 0$, by

$$(2.1) \qquad H^m(E) = \{v \mid D^j v \in L^2(E), \quad j = 0,\ldots,m\}$$

where D^j denotes d^j/dx^j. $H^m(E)$ has the usual Sobolev inner product and norm

$$(u,v)_{H^m(E)} = \sum_{j=0}^{m} (D^j u, D^j v)_{L^2(E)};$$

$$\|u\|_{H^m(E)} = [(u,u)_{H^m(E)}]^{\frac{1}{2}},$$

where

$$(\alpha,\beta)_{L^2(E)} = \int_E \alpha(x)\overline{\beta(x)}dx, \qquad \alpha,\beta \in L^2(E).$$

In the sequel, we use (u,v) instead of $(u,v)_{L^2(I)}$ and $\|u\|_m$ instead of $\|u\|_{H^m(I)}$.
The subspace $H_0^1(I)$ of $H^1(I)$ is defined by

(2.3) $$H_0^1(I) = \{v \mid v \in H^1(I); \quad v(0) = v(1) = 0\}.$$

Next we define the bilinear functional $B: H_0^1(I) \times H_0^1(I) \to \mathbb{C}$ associated with the operator L, L defined by (1.1), as follows:

(2.4) $$B(u,v) = (pDu,Dv) + (qu,v); u,v \in H_0^1(I).$$

We assume that $p(x)$ and $q(x)$ are such that B is strongly coercive, i.e.

(2.5) $$B(u,u) \geq C\|u\|_1^2, \qquad u \in H_0^1(I),$$

where C is independent of u.
A sufficient condition is $p(x) \geq p_0 > 0$, $q(x) \geq q_0 > -p_0\pi^2$ (see CIARLET, SCHULTZ & VARGA [1967]).

2.2 Galerkin's Method

Let $u: J \to H_0^1(I) \cap H^2(I)$ be the solution of (1.1). Then, as is well-known (see e.g. STRANG & FIX [1973], THOMÉE [1974]), u can be approximated by a $U: J \to S$, where S is some suitable finite-demensional subspace of $H_0^1(I)$. This U is given by the equation

$$\left(\frac{\partial}{\partial t} U(t,),V\right) + B(U(t,),V) = 0, \qquad V \in S,\ t \geq 0,\ U(0,) = U_0 \in S,$$

where U_0 is some suitable approximation to u_0.
For S we select the following subspace.
Let Δ: $0 = x_0 < x_1 < \ldots < x_{N-1} < x_N = 1$ be a uniform partition of I, i.e. $x_j = jh = jN^{-1}$, $j = 0,\ldots,N$. By I_j we denote the segment $[x_{j-1},x_j]$. We define the space $M_0^r(\Delta)$ (r a constant positive integer) by

(2.7) $$M_0^r(\Delta) = \{V \mid V \in H_0^1(I); V \in P_r(I_j), \quad j = 1,\ldots,N\},$$

where for any interval $E \subset I$ $P_\ell(E)$ denotes the space of polynomials of degree $d \le \ell$ restricted to E.

LEMMA 1. *Let* $U_0 \in M_0^r(\Delta)$ *be an approximation to* U_0 *satisfying*

(2.8) $$\|u_0-U_0\| = O(h^{r+1})$$

and let let $U: J \to M_0^r(\Delta)$ *satisfy* (2.6) *for all* t *with* U_0 *as initial function. Then the error function* $e(t,) = u(t,)-U(t,)$ *has the* L^2 *bound*

(2.9) $$\|e(t,)\|_0 \le Ch^{r+1}\|u(t,)\|_{r+1}, \quad \text{C independent of h.}$$

PROOF. STRANG & FIX [1973] give a proof for the case that U_0 is the elliptic projection of u_0, i.e.

(2.10) $$B(u_0-U_0,V) = 0, \ V \in M_0^r(\Delta).$$

Now, let U_0^* be any other approximation to u_0 satisfying (2.8) and let $U^*: J \to M_0^r(\Delta)$ be the solution of (2.6) with U_0^* as the initial function. Then it is easily proved that for $\varepsilon(t,) = U^*(t,) - U(t,)$ the relation

$$\left(\frac{\partial\varepsilon}{\partial t}(t,),V\right) + B(\varepsilon(t,),V) = 0, \ V \in M_0^r(\Delta)$$

holds. By putting $V = \varepsilon(t,)$, one proves that $\frac{d}{dt}\|\varepsilon(t,)\|_0^2$ is monotonically decreasing (B is strongly coercive) and hence

(2.11) $$\|U^*(t,) - U(t,)\|_0 \le \|U_0^*-U_0\|_0.$$

Since both $\|u_0-U_0\|_0$ and $\|u_0-U_0^*\|_0$ are of $O(h^{r+1})$, (2.11) implies that

$$\|u(t,)-U^*(t,)\|_0 \le \|u(t,)-U(t,)\|_0 + \|U(t,)-U^*(t,)\|_0 \le$$
$$\le Ch^{r+1} + \|U_0-U_0^*\|_0 = O(h^{r+1}),$$

which proves (2.9) for any approximation to u_0 of order h^{r+1}. □

3. SUPERCONVERGENCE AT THE KNOTS

As in the case of the two-point boundary value problems (see DOUGLAS et al. [1972,1974b]), the order of convergence at the knots is much higher than the global order of convergence, namely $O(h^{2r})$ vs. $O(h^{r+1})$. DOUGLAS et al. [1973,1974a,1974c]) gave proofs for several continuous time Galerkin methods. In this §, we intend to give a proof based on the use of the Laplace transform combined with the superconvergence results on two-point boundary value problems (see also CERRUTTI & PARTER [1976]).

For any v: $J \to V$, V a space of functions defined on I, and for any complex number s with positive real part, we define the function $v_s \in V$ by

$$(3.1)\qquad v_s(x) = \int_0^\infty e^{-st} v(t,x)\,dt, \qquad x \in I.$$

In imitation of STRANG & FIX [1973], we use s as a subscript, since it serves as a parameter only.
Next, we arrive at

THEOREM 1. *Let* $U_0 \in M_0^r(\Delta)$ *be the approximation to* U_0 *defined by*

$$(u_0-U_0,V) = 0, \qquad V \in M_0^r(\Delta)$$

and let $U\colon J \to M_0^r(\Delta)$ *be the solution to* (2.6) *with* U_0 *as the initial function. Then the error function* $e(t,x) = u(t,x) - U(t,x)$ *has the global bound* (2.9) *and the pointwise bound*

$$(3.3)\qquad |e(t,x_i)| = O(h^{2r}), \qquad t \ge 0, \qquad i = 1,\dots,N-1.$$

PROOF. For any $V \in M_0^r(\Delta)$ we have

$$\|U_0-V\|_0^2 = (u_0-V,U_0-V) + (U_0-u_0, U_0-V) =$$
$$= (u_0-V,U_0-V) \leq \|u_0-V\|_0 \|U_0-V\|_0.$$

and hence $\|U_0-V\|_0 \leq \|u_0-V\|_0$, so it follows that for any V

$$\|u_0-U_0\|_0 \leq \|u_0-V\|_0 + \|U_0-V\|_0 \leq 2\|u_0-V\|_0.$$

The last inequality implies that $\|u_0-U_0\|_0 = O(h^{r+1})$, hence Lemma 1 can be applied to prove the global error bound (2.9).

In order to prove the pointwise error bound, we consider the Laplace transform $u_s \in H_0^1(I) \cap H^2(I)$ of the solution u of (1.1). This u_s is the solution of the two-point boundary value problem (in Galerkin form)

$$B(u_s,v) + s(u_s,v) = (u_0,v), \quad v \in H_0^1(I) \tag{3.4}$$

and can be approximated in $M_0^r(\Delta)$ by the solution U_s of

$$B(U_s,V) + s(U_s,V) = (u_0,V), \quad V \in M_0^r(\Delta). \tag{3.5}$$

Now, on the other hand, let us consider the Laplace transform U_s^* of the solution U of (2.6), where U_0 defined by (3.2) is the initial function. Analogue to (3.3), U_s^* is the solution of

$$B(U_s^*,V) + s(U_s^*,V) = (U_0,V), \quad V \in M_0^r(\Delta). \tag{3.6}$$

If we subtract (3.6) from (3.5), take (3.2) into account and substitute $V = U_s-U_s^*$, we obtain for any s

$$B(U_s-U_s^*,U_s-U_s^*) + s\|U_s-U_s^*\|_0^2 = 0. \tag{3.7}$$

Since Re $s > 0$, we may write $s = \sigma + i\tau$, $\sigma > 0$.

If $\tau \neq 0$, then $\|U_s - U_s^*\|_0 = 0$, otherwise,

$$0 = B(U_s - U_s^*, U_s - U_s^*) + \sigma \|U_s - U_s^*\|_0^2$$
$$\geq B(U_s - U_s^*, U_s - U_s^*) \geq C\|U_s - U_s^*\|_1^2$$

hence $U_s \equiv U_s^*$ for all values of s with $\mathrm{Re}\ s > 0$.

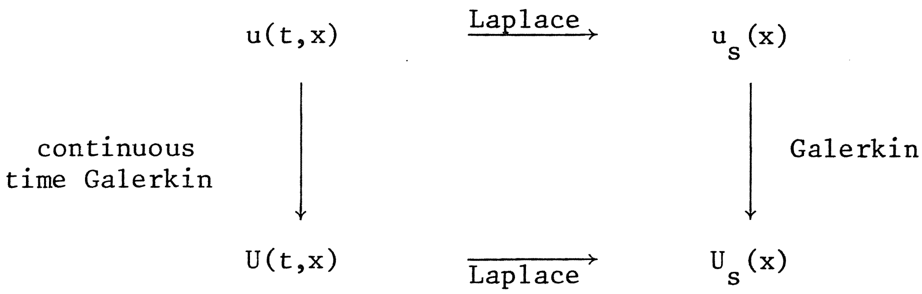

Figure 1.

From Douglas et al. [1974b] it is known that the error of function $e_s(x) = u_s(x) - U_s(x)$ has the local bound at the knots

(3.8) $\quad |e_s(x_i)| = O(h^{2r}), \qquad i = 1,\ldots,N-1.$

This implies that $e_s(x_i)$ can be written as $e_s(x_i) = E_s(x_i;h)h^{2r}$, where $|E_s(x_i;h)|$ has an upper bound independent of h, as $h \to 0$. Now suppose that $e(t,x_i) = E(t,x_i;h)h^{2r}$. Then $E_s(x_i;h)$ is the Laplace transform of $E(t,x_i;h)$. Since E_s has an upper bound independent of h, as $h \to 0$, it turns out that $E(t,x_i;h)$ has an upper bound independent of h, as $h \to 0$. This proves (3.3). □

<u>REMARK</u>. Note that the use of the Laplace transform succeeded because the Laplace transformation and differentiation with respect to x were interchangeable. This interchangeability is a result of the stationary nature of the problem (1.1): the operator L is independent of t.

4. QUADRATURE RULES

In practice, one is forced to evaluate B(U,V) by some quadrature rule in order to generate the initial value problem (see RAVIART [1972]). In this § we advocate the use of a special quadrature rule because it has the following advantages:

1. there is no loss of accuracy, neither global nor pointwise;
2. a purely explicit initial value problem can be obtained by selecting a proper basis of $M_0^r(\Delta)$.

4.1. Lobatto quadrature

It is known (cf. e.g. DAVIS & RABINOVITZ [1967]) that if $f \in C^{2r}(I)$, the integral

$$\int_0^1 f(\sigma)d\sigma$$

can be approximated by the (r+1)-point Lobatto quadrature

$$Q[f] = \sum_{\ell=0}^{r} w_\ell f(\sigma_\ell), \tag{4.1}$$

where $\sigma_0 = 0$, $\sigma_r = 1$ and $\sigma_1,\ldots,\sigma_{r-1}$ are distinct points inside I. The weights w_ℓ are positive. Examples are the trapezium rule $(r = 1)$ and Simpson'r ule $(r = 2)$. The approximation (4.1) is exact when $f \in P_{2r-1}(I)$, otherwise the error is of $O(D^{2r}f(\xi),\ \xi \in (0,1))$, or of $O(\|D^{2r}f\|_0)$. Next, we define $(\alpha,\beta)_h$ by

$$\begin{aligned}(\alpha,\beta)_h &= \sum_{j=1}^{N} (\alpha,\beta)_j^* ;\\ (\alpha,\beta)_j^* &= h\sum_{\ell=0}^{r} w_\ell\, \alpha(\xi_{j,\ell})\beta(\xi_{j,\ell}), \qquad j = 1,\ldots,N;\\ \xi_{j,\ell} &= x_{j-1} + h\sigma_\ell; \quad j = 1,\ldots,N;\ \ell = 0,\ldots,r.\end{aligned} \tag{4.3}$$

Note that $\xi_{j+1,0} = \xi_{j,r}$. Now $(\alpha,\beta)_h = (\alpha,\beta)$ if $\alpha\beta \in P_{2r-1}(I_j)$, $j = 1,\dots,N$; otherwise if $\alpha\beta \in H^{2r}(I_j)$

$$(4.4) \qquad |(\alpha,\beta)_h-(\alpha,\beta)| = O(h^{2r}) \sum_{j=1}^{N} \|\alpha\beta\|_{H^{2r}(I_j)}.$$

Inequality (4.4) can be proved by direct application of the lemma of BRAMBLE & HILBERT [1970, p.114].

LEMMA 2. *For any* $U \in M_0^r(\Delta)$

$$|U|_h = [(U,U)_h]^{\frac{1}{2}}$$

is a norm equivalent to $\|U\|_0$.

PROOF. It is sufficient to prove that for any I_j $(U,U)_j^*$ is equivalent to $(U,U)_{L^2(I_j)}$. To that end, let $f \in P_r(I)$ be represented by

$$(4.5a) \qquad f(\sigma) = \sum_{j=0}^{r} a_j L_j(\sigma)$$

or by

$$(4.5b) \qquad f(\sigma) = \sum_{j=0}^{r} f(\sigma_j)\phi_j(\sigma),$$

where $\phi_j(\sigma)$, $j = 0,\dots,1$ are the rth degree Lagrange interpolation polynomials chosen such that $\phi_j(\sigma_i) = \delta_{ij}$ and where $L_j(\sigma)$, $j = 0,\dots,r$ is the j-th degree normalized Legendre polynomial shifted to the interval I. Now it turns out that, by the orthonormality of $\{L_j\}_{j=0}^{r}$,

$$(4.6a) \qquad \|f\|_0^2 = \sum_{j=0}^{n} a_j^2 \|L_j\|_0^2 = \sum_{j=0}^{n} a_j^2 \;;$$

$$(4.6b) \qquad Q[f^2] = \sum_{j=0}^{n} [f(\sigma_j)]^2 w_j$$

Since both representation (4.5) of f are equivalent, it easily follows that $Q[f^2]$ is a norm on $P_r(I)$ equivalent to $\|f\|_0^2$. Now since,

$$(U,U)^*_j = h \sum_{\ell=0}^{r} w_\ell [U(h\sigma_\ell + x_{j-1})]^2;$$

$$(U,U)_{L^2(I_j)} = h \int_0^1 [U(h\sigma + x_{j-1})]^2 d\sigma$$

the equivalence of $(U,U)^*_j$ and $(U,U)_{L^2(I_j)}$ is proved by substituing $f(\sigma) = U(h\sigma + x_{j-1})$. This proves the equivalence of $|U|_h$ and $\|U\|_0$. □

Next, we define the bilinear functional $B_h\colon M_0^r(\Delta) \times M_0^r(\Delta) \to \mathbb{C}$ by

$$B_h(U,V) = (pU',V')_h + (qU,V)_h, \quad U,V \in M_0^r(\Delta). \tag{4.7}$$

LEMMA 3. *For sufficiently small* h, *the inequalities*

$$\begin{aligned} &|B_h(U,V) - B(U,V)| = O(h^{\ell+m}\|U\|_{\ell,\Delta}\|V\|_{m,\Delta}); \\ &|(U,V)_h - (U,V)| = O(h^{\ell+m}\|U\|_{\ell,\Delta}\|V\|_{m,\Delta}); \quad U,V \in M_0^r(\Delta) \\ &\qquad 0 \le \ell, m \le r, \end{aligned} \tag{4.8}$$

hold, where $\|U\|_{m,\Delta}$ *is defined by*

$$\|U\|_{m,\Delta} = [\sum_{j=1}^{N} \|U\|^2_{H^m(I_j)}]^{\frac{1}{2}}, \quad U \in M_0^r(\Delta). \tag{4.9}$$

Note that $\|U\|_{m,\Delta} = \|U\|_m$ *if* $m = 0,1$.

PROOF. We only give a proof of the former inequality

$$\begin{aligned} B_h(U,V) - B(U,V) &= O(h^{2r} \sum_{j=1}^{N} \|D^{2r}(pU'V'+qUV)\|_{L^2(I_j)}) = \\ &= O(h^{2r} \sum_{j=1}^{N} \|U\|_{H^r(I_j)}\|V\|_{H^r(I_j)}) = O(h^{2r}\|U\|_{r,\Delta}\|V\|_{r,\Delta}) = \\ &= O(h^{\ell+m}\|U\|_{\ell,\Delta}\|V\|_{m,\Delta}). \end{aligned}$$

The last step is proved by HEMKER [1977]. Analogously, the second bound is proved. After these technical lemmas, we arrive at

Theorem 2. *Let* $U_0 \in M_0^r(\Delta)$ *br the approximation to* $u_0 \in H_0^1(I)$ *defined by*

(4.10) $$(u_o-U_0,V)_h = 0, \; V \in M_0^r(\Delta)$$

and let $Y : J \to M_0^r(\Delta)$ *be the solution of*

(4.11) $$\left(\frac{\partial Y}{\partial t},V\right)_h + B_h(Y,V) = 0, \; V \in M_0^r(\Delta), \; Y(0,) = U_0.$$

Then the error function $e(t,x) = u(t,x) - Y(t,x)$ had the following bounds

(4.12) $$\|e(t,)\|_0 = O(h^{r+1});$$
$$|e(t,x_i)| = O(h^{2r}).$$

Note that (4.10) implies $U_0(\xi_{j,\ell}) = u_0(\xi_{j,\ell})$, $j = 1,\ldots,N$; $\ell = 0,\ldots,r$.

PROOF. Let $U:J \to M_0^r(\Delta)$ be the solution of (2.6) with U_0, defined by (4.10), as initial function. We know that $\|u_0-U_0\|_0 = O(h^{r+1})$, since (4.10) leaves elements of $M_0^r(\Delta)$ invariant (see CIARLET & RAVIART [1972], hence $\|u(t,) - U(t,)\|_0 = O(h^{r+1})$, by lemma 1. Now we define

(4.13) $$\varepsilon\,(t,x) = U(t,x) - Y(t,x).$$

By (2.6), (4.11) and lemma 3, we obtain

(4.14) $$\left(\frac{\partial \varepsilon}{\partial t},V\right)_h + B_h(\varepsilon,V) = B_h(U,V) - B(U,V) + (U,V)_h - (U,V)$$
$$= O(h^{r+1}\|V\|_1(\|U_t\|_{r,\Delta} + \|U\|_{r,\Delta})).$$

Since $\|u-U\|_0$ is of $O(h^{r+1})$, it follows that $\|U_t\|_{r,\Delta}$ and $\|U\|_{r,\Delta}$ are bounded by $\|u_t\|_r$ and $\|u\|_r$ (see CIARLET & RAVIART [1972]). Hence, if we fill in $V = \varepsilon(t,)$, we obtain

(4.15) $$\left(\frac{\partial \varepsilon}{\partial t}(t,), \varepsilon(t,)\right)_h + B_h(\varepsilon(t,), \varepsilon(t,)) = O(h^{r+1}\|\varepsilon(t,)\|_1(\|u\|_r+\|u_t\|_r).$$

Since B_h is strongly coercive for sufficiently small h (set $\ell = m = 1$ in (4.8)), it follows from (4.15) that

(4.16) $$\frac{d}{dt} |\varepsilon(t,)|_h^2 + \|\varepsilon(t,)\|_1^2 \le C\, h^{r+1} \|\varepsilon(t,)\|_1 .$$

If we apply Gronwell's inequality we obtain

$$\frac{d}{dt} |\varepsilon(t,)|_h^2 \le C\, h^{2r+2}$$

Let us set $|\varepsilon(t,)|_h = F(t)h^{r+1}$, then F(t) satisfies the differential inequality

(4.17) $$2F \frac{dF}{dt} \le C; \; F(0) = 0.$$

(4.17) implies that F(t) is bounded by $C\sqrt{t}$ and hence has a bound independent of h, so $|\varepsilon(t,)|_h \le C\, h^{r+1}$. Now the first bound is proved by

$$\|u(t,)-Y(t,)\|_0 \le \|u(t,)-U(t,)\|_0 + \|\varepsilon(t,)\|_0 \le$$

$$\le \|u(t,)-U(t,)\|_0 + C|\varepsilon(t,)|_h = O(h^{r+1}).$$

The second bound is proved in exactly the same way as was done in theorem 1. Again, let $u_s \in H_0^1(I)$ be the Laplace transform of u(t,) satisfying (3.4) and let Y_s be the Laplace transform of Y(t,) which satisfies the relation

(4.19) $$B_h(Y_s,V) + s(Y_s,V)_h = (U_0,V)_h, \quad V \in M_0^r(\Delta).$$

Now, since $(u_0,V)_h = (U_0,V)_h$, Y_s is a Galerkin solution of (3.4), where (r+1)-point Lobatto quadrature is used (see HEMKER [1975]). As HEMKER [1975] and DOUGLAS et al. [1974b] pointed out, the error function $u_s(x) - Y_s(x)$ has the pointwise bound

$$|u_s(x_i) - Y_s(x_i)| \le C\, h^{2r}; \quad i = 1,\dots,N-1,$$

where C does not depend on h. Now let

$$u_s(x_i) - Y_s(x_i) = E_s(x_i;h)h^{2r};$$

$$u(t,x_i) - Y(t,x_i) = E(t,x_i;h)h^{2r}.$$

Then $E_s(x_i;h)$ is the Laplace transform of $E(t,x_i;h)$. Since $|E_s(x_i;h)|$ has an upper bound independent of h, it turns out that $|E(t,x_i;h)|$ has an upper bound independent of h. This proves the latter part of (4.12). □

4.2 Purely explicit initial value problems

HEMKER [1975] proves that the basis $\{\phi_i\}_{i=1}^{rN-1}$ of $M_0^r(\Delta)$ can be constructed such that

$$(\phi_i,\phi_j)_h = \delta_{ij}(\phi_i,\phi_i)_h = \delta_{ij}\ \lambda_i; \qquad i,j = 1,\ldots,rN-1.$$

If we represent Y(t,x) by

$$Y(t,x) = \sum_{j=1}^{rN-1} a_j(t)\phi_j(x)$$

and apply (4.11) for ϕ_i, i = 1,...,rN-1, we obtain

$$\lambda_i \frac{da_i}{dt} + \sum_{j=1}^{rN-1} B_h(\phi_j,\phi_i)a_j = 0,\ i = 1,\ldots,rN-1; \qquad t \geq 0$$

$$Y(0,\xi_{j,\ell}) = u_0(\xi_{j,\ell}).$$

So the use of Labatto quadrature makes it possible to obtain purely explicit initial value problems which may be integrated by explicit methods

5. Numerical Example

In order to illustrate the superconvergence at the knots when Lobatto quadrature is applied, we integrated the following simple problem

$$\frac{\partial u}{\partial t} = 2\,\frac{\partial^2 u}{\partial x^2} + (x^{10}+180x^8-x)e^{-t}, \qquad t \geq 0;$$

$$u = 0, \qquad x = 0,\ 1; \qquad u = x - x^{10}, \qquad t = 0.$$

the exact solution is $(x-x^{10})e^{-t}$. I was partitioned into N segments of equal length for N = 4, 8, 16 respectively. For r = 1, 2, 3 this problem was integrated from 0 to 1 by an adaptive Runge Kutta method. Below, we list the errors in the points 0.25, 0.50 and 0.75 for r = 1, 2, 3 and N = 4, 8, 16.

r = 1

x \ N	4	8	16
0.25	3.90(-2)	1.11(-2)	2.87(-3)
0.50	7.65(-2)	2.17(-2)	5.61(-3)
0.75	9.96(-2)	2.80(-2)	7.20(-3)

r = 2

x \ N	4	8	16
0.25	1.87(-3)	1.25(-4)	7.97(-6)
0.50	3.61(-3)	2.40(-4)	1.53(-5)
0.75	4.25(-3)	2.80(-4)	1.77(-5)

r = 3

x \ N	4	8	16
0.25	1.15(-5)	1.83(-7)	2.78(-9)
0.50	2.04(-5)	3.23(-7)	4.91(-9)
0.75	2.01(-5)	3.17(-7)	4.79(-9)

One easily verifies that the errors decrease by about 2^{-2r}, when N is doubled, which confirms the superconvergence at the knots.

LITERATURE

BRAMBLE, J.H. & S.R. HILBERT, *Estimation of linear functionals on Sobolev spaces with application to Fourier transforms and spline interpolation*, SIAM J. Num. An. 7 (1970), pp. 112 - 124.

CIARLET, P.G. & P.A. RAVIART, *General Lagrange and Hermite interpolation in* R^N *with applications to finite element methods*, Arch. Rat. Mech. An. 46 (1972), pp. 177 - 199.

CIARLET, P.G., M.H. SCHULTZ & R.S. VARGA, *Numerical methods of high-order accuracy for nonlinear boundary value problems I: One-dimensional problems*, Num. Math. 9 (1967), pp. 394 - 430.

CERRUTTI, J.H. & S.V. PARTER, *Collocation methods for parabolic partial differential equations in one space dimension*, Num. Math. 26 (1976), pp. 227 - 254.

DAVIS, P.J. & P. RABINOWITZ, *Numerical integration*, Blaisdell, New York - Toronto - London, 1967.

DOUGLAS, J., jr. & T. DUPONT, *Some superconvergence results for Galerkin methods for the approximate solution of two-point boundary problems*, from MILLER, Topics in Numerical Analysis.

______ , ______ & M.F. WHEELER, *Some superconvergence results for an* H^1*-Galerkin precedure for the heat equation*, Report MRC 1382, Madison, Wisconsin 1973.

______ , ______ & ______, *A quasi-projection approximation method applied to Galerkin procedures for parabolic and hyperbolic equstions*, Report MRC 1461, Madison, Wisconsin, 1974.

______ & ______, *Galerkin approximations for the two-point boundary problem using continuous, piecewise polynomial spaces*, Num. Mat. 22 (1974), pp. 99 - 109.

______ & ______, *Collocation methods for parabolic equations in a single space variable*, Springer-Verlag, Heidelberg, 1974.

HEMKER, P.W., *Galerkin methods and Lobatto points*, MC report 24/75, Mathematisch Centrum, Amsterdam, 1975.

HEMKER, P.W., *A numerical study of stiff two-point boundary problems*, MC tract 80, Mathematisch Centrum, Amsterdam, 1977.

MILLER, J.J.H. (ed.), *Topics in numerical analysis*, Academic Press, London-New-York, 1973

RAVIART, P.A., *The use of numerical integration in finite element methods for solving parabolic equations*, From: J.J.H. Miller (ed.), Topics in Numerical Analysis.

STRANG, G. & G.J. FIX, *An analysis of the finite element method*, Prentice-Hall, Englewood Cliffs, New Jersey, 1973.